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THE GRAVITY FIELD FOR AN ELLIPSOID OF REVOLUTION AS A LEVEL SURFACE

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NOTICES

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FOREWORD

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ABSTRACT

The form of a single level surface that envelops all attracting matter and the value of the potential or of the gravitational attraction at a definite point on or outside the surface determine uniquely the field of force on and outside of the surface. This statement holds good even when we add to the gravitational attraction the centrifugal "force" due to uniform rotation. Hence, if we assume that the surface of the earth is an exact ellipsoid of rotation under its own attraction combined with the centrifugal force of uniform rotation about the minor axis, the field of force on and outside this ellipsoid is uniquely determined, if we assume, for instance, the value of gravity at the equator.

This determination is carried out by means of a rather special set of curvilinear coordinates. In terms of these coordinates the value of the potential at any point in exterior space and the value of gravity are expressible in closed form in terms of Lengendre functions of the second kind with imaginary argument, and these again are expressible in terms of the elementary functions. In practice, however, it is found more convenient to expand the Lengendre functions in series, also the formulas dependent on them. See formula (32)p.12 for the potential expressed in closed form, and formulas (45),p.15, and formulas (61) and (63),p.20 also formulas (76)p.23 for the values of gravity on the ellipsoid.

Various incidental developments are given. This is first of a series of at least three papers on the general subject.

PUBLICATION REVIEW

FOR THE COMMANDING GENERAL:

E.L. Woodford
for DELWIN B. AVERY
Colonel, USAF
Chief, Photo Reconnaissance Lab.
Weapons Components Division

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THE GRAVITY FIELD FOR AN ELLIPSOID OF REVOLUTION AS A LEVEL SURFACE

INTRODUCTION

In Volume II of Stokes' collected papers 1/ there are two mathematical articles dealing with attractions and gravity, both of them published in 1849. In the first of these, published in the Cambridge and Dublin Mathematical Journal and entitled "On Attractions and Clairaut's Theorem," Stokes states several theorems, disclaiming, as he says, any pretense to originality. Many of the theorems are in fact due to Gauss.

The second paper, published in the Transactions of the Cambridge Philosophical Society and entitled "On the Variation of Gravity at the Surface of the Earth," gives the formula 2/ on which the whole gravity project of the Laboratory ultimately depends. In the opening pages of this second paper are two passages that seem worth quoting as an introduction to this paper. They deal with the general problem of this paper rather than with the details of Stokes' Formula. For the present purpose the two concluding sentences of the first passage are the important ones.

1/ George G. Stokes, Mathematical and Physical Papers, Vol. 2, Cambridge, 1883.

2/ It is convenient to distinguish between Stokes' Formula given in this paper of 1849 and what is commonly known as Stokes' Theorem, a very general mathematical theorem dealing with the line integral taken around a closed curve of the tangential component of a vector point function and its relation to a certain surface integral.

"On adopting the hypothesis of the earth's original fluidity, it has been shewn that the surface ought to be perpendicular to the direction of gravity, that it ought to be of the form of an oblate spheroid of small ellipticity, having its axis of figure coincident with the axis of rotation, and that gravity ought to vary along the surface according to a simple law, leading to the numerical relation between the ellipticity and the ratio between polar and equatorial gravity which is known by the name of Clairaut's Theorem. Without assuming the earth's original fluidity, but merely supposing that it consists of nearly spherical strata of equal density, and observing that its surface may be regarded as covered by a fluid, inasmuch as all observations relating to the earth's figure are reduced to the level of the sea, Laplace has established a connexion between the form of the surface and the variation of gravity, which in the particular case of an oblate spheroid agrees with the connexion which is found on the hypothesis of original fluidity. The object of the first portion of this paper is to establish this general connexion without making any hypothesis whatsoever respecting the distribution of matter in the interior of the earth, but merely assuming the theory of universal gravitation. It appears that if the form of the surface be given, gravity is determined throughout the whole surface, except so far as regards one arbitrary constant which is contained in its complete expression, and which may be determined by the value of gravity at one place. Moreover the attraction of the earth at all external points of space is determined at the same time; so that the earth's attraction on the moon, including that part of it which is due to the earth's oblateness, and the moments of the forces of the sun and moon tending to turn the earth about an equatorial axis, are found quite independently of the distribution of matter within the earth."

With respect to the determination of the gravity field from a knowledge of the form of a level surface and the value of gravity at a definite point on the surface or at a definite external point Stokes remarks:

"Nevertheless, although we know that the problem is always determinate, it is only for a very limited number of forms of the surface S that the solution has hitherto been effected. The most important of these forms is the sphere. When S has very nearly one of these forms the problem may be solved by approximation."

In the case of an exact ellipsoid no approximation is needed. Stokes had already noted the purely mathematical analogy of the problem to a seemingly very different physical problem, that of the steady state of temperatures in a homogeneous solid, subject to certain boundary conditions. The solution in this Part I is based on one of the commonest methods of attacking these problems in the flow of heat. The second solution, a synthetic one, based on the concept of "coating," is given in Part II, partly as an illustration of the convenience of this concept, partly because of certain other advantages. Part III will give some further developments.

§1 - The Special Ellipsoidal Coordinates and their Properties.

The complete development of the subject would occupy an undue amount of space. For the proofs of many of the statements reference may be made to books such as Byerly's Fourier's Series and Spherical Harmonics, (Boston, 1895), or Todhunter's The Functions of Laplace, Lame and Bessel. (London, 1875). Detailed discussions of the Legendre functions and of Laplace's equation in curvilinear coordinate systems may also be found in Whittaker and Watson's Modern Analysis (Cambridge, 1927) or MacRobert's Spherical Harmonics (London, 1927).

If the z-axis be taken as the axis of rotation, the relation between ordinary rectangular coordinates x, y and z and the special ellipsoidal coordinates α, β and λ is taken as

$$\begin{aligned} x &= c \operatorname{cosec} \alpha \operatorname{Sech} \beta \cos \lambda \\ y &= c \operatorname{cosec} \alpha \operatorname{Sech} \beta \sin \lambda \\ z &= c \cot \alpha \operatorname{Tanh} \beta. \end{aligned} \tag{1}$$

These formulas are those of equation (12) p. 242 of Byerly with α replaced by $\frac{\pi}{2} - \alpha$, in order that the α here used may remain small in practice. Byerly's y and z are here interchanged.

From (1) we find

$$\operatorname{Sech}^2\beta + \operatorname{Tanh}^2\beta = 1 = \frac{x^2}{c^2\operatorname{cosec}^2\alpha} + \frac{y^2}{c^2\operatorname{cosec}^2\alpha} + \frac{z^2}{c^2\cot^2\alpha}. \tag{2}$$

The family of surfaces α = const. thus represents a set of confocal ellipsoids of revolution about the z-axis, the distance from center to focus being

$$\sqrt{c^2\operatorname{cosec}^2\alpha - c^2\cot^2\alpha} = c.$$

Equation (2) will give α in terms of x, y and z. The equation to be solved is a quadratic in the square of a trigonometric function; only one root of the quadratic is available; the other root gives imaginary values of α. The result of solving for α may be written

$$\left.\begin{aligned} \sin^2\alpha &= \frac{r^2+c^2-\sqrt{(r^2+c^2)^2-4c^2(x^2+y^2)}}{2(x^2+y^2)}, \\ \text{or} \qquad & \\ \cot^2\alpha &= \frac{r^2-c^2+\sqrt{(r^2-c^2)^2+4c^2z^2}}{2c^2}. \end{aligned}\right\} \qquad (3)$$

In (3) r^2 stands for $x^2+y^2+z^2$.

In a similar way

$$\operatorname{cosec}^2\alpha - \cot^2\alpha = 1 = \frac{x^2}{c^2\operatorname{Sech}^2\beta} + \frac{y^2}{c^2\operatorname{Sech}^2\beta} - \frac{z^2}{c^2\operatorname{Tanh}^2\beta}. \qquad (4)$$

The family of surfaces β = const. thus represents a set of unparted confocal hyperboloids of revolution, the distance from center to focus being $\sqrt{c^2\operatorname{Sech}^2\beta + c^2\operatorname{Tanh}^2\beta} = c$, as before. The ellipsoids are therefore confocal with the hyperboloids and consequently the two families of surfaces intersect each other orthogonally.

From the first two equations of (1)

$$\frac{y}{x} = \tan\lambda \qquad (5)$$

which, for λ = const., represents a family of planes containing the z-axis. These planes evidently cut both the ellipsoids and the hyperboloids orthogonally.

Any point in space, therefore, may be given by specifying the values of α, β, and λ; these numbers determine the ellipsoid, hyperboloid and plane, the intersection of which fixes the point. There are apparently four points of intersection of the three surfaces, but the ambiguity may be removed by using the analogy between the quantities β and λ and ordinary latitude and longitude. It is evident from (4) that λ is indeed the longitude of the point reckoned from the xz-plane and if, instead of the entire plane defined by (4), we use only the half-plane on the side of the axis specified by the angle λ, the four possible points of intersection are reduced to two, one above and one below the xy-plane.

The relation of the coordinate β to latitude may be derived by considering the well-known parametric equations of an ellipse of semi-axes a and b, in the xz-plane

$$\left.\begin{aligned} x &= a \cos u \\ z &= b \sin u \end{aligned}\right\} \tag{6}$$

The geometric significance of the parameter u and its relation to the vectorial angle Ψ is shown in figure 1. The geometric relations there indicated suggest the name "reduced latitude" frequently given to u, while by astronomical analogy Ψ is the geocentric latitude and Φ is the geographic latitude. The well-known relations between these latitudes are

$$\left.\begin{aligned} \tan u &= \frac{a}{b} \tan \Psi = \frac{b}{a} \tan \Phi, \\ \tan \Psi &= \frac{b}{a} \tan u = \frac{b^2}{a^2} \tan \Phi, \\ \tan \Phi &= \frac{a}{b} \tan u = \frac{a^2}{b^2} \tan \Psi. \end{aligned}\right\} \tag{7}$$

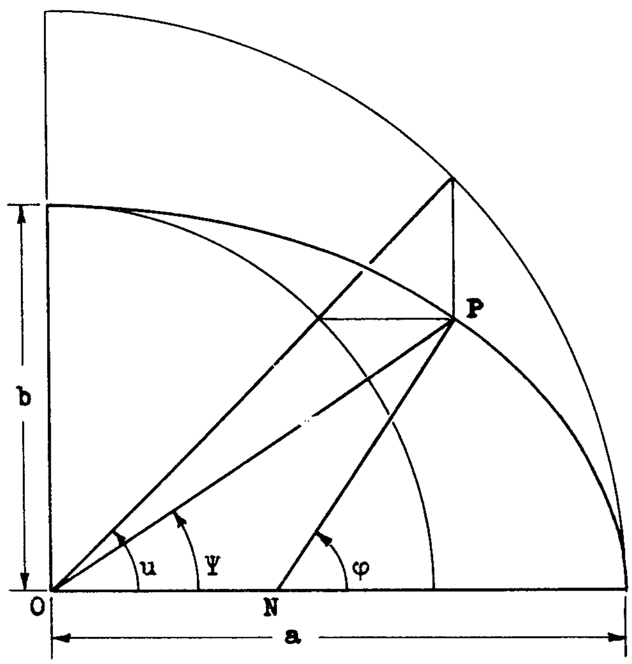

Figure 1 - Meridional Section of an Ellipsoid

a = semimajor axis,

b = semiminor axis,

PN is normal to the ellipse, φ being the geographic latitude of P.

The vectorial angle Ψ is the geocentric latitude of P.

The angle u is usually referred to as the reduced latitude of P.

From (2) it is seen that the semi-axes of the ellipsoid are $c \operatorname{cosec} \alpha$ and $c \cot \alpha$. By writing these for a and b in (6) and introducing the additional coordinates y and λ to make the ellipse (6) into an ellipsoid of revolution about the axis we get

$$\left.\begin{aligned} x &= c \operatorname{cosec} \alpha \cos u \cos \lambda, \\ y &= c \operatorname{cosec} \alpha \cos u \sin \lambda, \\ z &= c \cot \alpha \sin u. \end{aligned}\right\} \qquad (8)$$

By comparing (8) and (1) it is seen that the two sets of equations would be identical if we put

$$\left.\begin{aligned} \operatorname{Sech} \beta &= \cos u, \\ \operatorname{Tanh} \beta &= \sin u. \end{aligned}\right\} \qquad (9)$$

The two equations (9) are equivalent and mean that the reduced latitude u is the Gudermannian angle of β. This implies besides (9) the additional relations

$$\left.\begin{aligned} \beta &= \log_e \tan \left(\frac{\pi}{4} + \frac{1}{2} u\right), \\ \operatorname{Sinh} \beta &= \tan u, \\ \operatorname{Cosh} \beta &= \sec u. \end{aligned}\right\} \qquad (10)$$

Equations (9) and (10) enable us to use either u or β in our equations at will. It is found more convenient to use β when obtaining expressions for the potential and the intensity of gravity, but after the results are obtained, it is more satisfactory to express them in terms of u. The fact that u and β have the same sign and that u may be considered as the reduced latitude enables us to choose between the two points of intersection of the ellipsoid given by equation (2), the hyperboloid given by equation (4) and the half-plane corresponding to the meridian of λ given by equation (5). The point on the positive side of

the xy-plane corresponds to a positive or north latitude and a point on the negative side to a negative or south latitude.

When x, y and z are given, α is found from (3); β is then found by the following formula easily deduced from (1)

$$\operatorname{Sinh} \beta = \tan u = z \frac{\sec \alpha}{\sqrt{x^2 + y^2}}, \tag{11}$$

and λ is found from (5).

Any expression for the potential V due to attracting matter must at points outside of such matter satisfy Laplace's equation. For the special ellipsoidal coordinates α, β and λ here adopted, Laplace's equation is

$$\sin^2 \alpha \frac{\partial^2 V}{\partial \alpha^2} + \operatorname{Cosh}^2 \beta \frac{\partial^2 V}{\partial \beta^2} + (\operatorname{Cosh}^2 \beta - \sin^2 \alpha) \frac{\partial^2 V}{\partial \lambda^2} = 0. \tag{12}$$

See Byerly, p. 242 equation (11). This equation may also be readily obtained from the equation given by Whittaker and Watson, p. 551. Special solutions of this equation are

$$\begin{aligned} V_{mn} &= (-i)^{m-n} (A_n \cos n\lambda + B_n \sin n\lambda)\, P_m^n(\operatorname{Tanh} \beta)\, P_m^n (i \cot \alpha), \\ V_{mn} &= (i)^{m+n+1} (A_n \cos n\lambda + B_n \sin n\lambda)\, P_m^n(\operatorname{Tanh} \beta) \sec^n \alpha \frac{d^n Q_m(i \cot \alpha)}{d(i \cot \alpha)^n}. \end{aligned} \tag{13}$$

In equations (13) A_n and B_n are constants,

$P_m(\operatorname{Tanh} \beta)$ represents a Legendre's zonal harmonic of degree _m_,

$P_m^n(\operatorname{Tanh} \beta)$ is the associated function $\operatorname{Sech}^n \beta \dfrac{d^n P_m(\operatorname{Tanh} \beta)}{d(\operatorname{Tanh} \beta)^n}$,

with a corresponding meaning for $P_m^n(i \cot \alpha)$.

$Q_m(i \cot \alpha)$ represents a Legendre's function of the second kind.

The expressions for Q_o (x) and Q_1 (x), x being any variable, are

$$\left.\begin{aligned} & Q_o(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \text{ if } |x| < 1, \\ \text{or} \qquad & Q_o(x) = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), \text{ if } |x| > 1. \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} & Q_1(x) = -1 + \frac{x}{2}\log_e\left(\frac{1+x}{1-x}\right), \text{ if } |x| < 1, \\ & Q_1(x) = -1 + \frac{x}{2}\log_e\left(\frac{x+1}{x-1}\right), \text{ if } |x| > 1. \end{aligned}\right\} \tag{15}$$

From Q_o (x) and Q_1 (x) the Q's of higher orders may be found by means of the recurrence formula

$$(m+1)\ Q_{m+1}(x) - (2m+1)x\ Q_m(x) + m\ Q_{m-1}(x) = 0 \tag{16}$$

The Q's are connected with the P's by the relation

$$Q_n(x) = \frac{1}{2}\ P_n(x)\ \log_e\left(\frac{1+x}{1-x}\right) - R, \tag{17}$$

where

$$R = \frac{2n-1}{1\ n} P_{n-1}(x) + \frac{2n-5}{3(n-1)} P_{n-3}(x) + \frac{2n-9}{5(n-2)} P_{n-5}(x) + \cdots$$

$$\cdots + \frac{2n-4k+3}{(2k-1)(n-k+1)} P_{n-2k+1}(x) + \cdots \tag{18}$$

The series for R ends with a term in P_1 (x) when n is even and with a term in P_o (x) if n is odd. If $|x| > 1$, $\log_e\left(\frac{1+x}{1-x}\right)$ in (17) is to be replaced by $\log_e\left(\frac{x+1}{x-1}\right)$

Another form for Q_m (x), valid when $|x| > 1$ is the infinite series

$$Q_m(x) = \frac{m!}{(2m+1)(2m-1)\cdots 3.1}\left[\frac{1}{x^{m+1}} + \frac{(m+1)(m+2)}{2\cdot(2m+3)}\frac{1}{x^{m+3}}\right.$$

$$\left. + \frac{(m+1)(m+2)(m+3)(m+4)}{2.4\ (2m+3)(2m+5)}\ \frac{1}{x^{m+5}} \cdots\right]. \tag{19}$$

From the definitions we find

$$
\begin{aligned}
Q_0(i \cot \alpha) &= -\, i\, \alpha, \\
Q_1(i \cot \alpha) &= -\, 1 + \alpha \cot \alpha, \qquad\qquad (20)\\
Q_2(i \cot \alpha) &= -\, i\left[\frac{3}{2} \cot \alpha - \left(\frac{3}{2} \cot^2 \alpha + \frac{1}{2}\right)\alpha\right].
\end{aligned}
$$

The derivatives of Q_0 and Q_2 will be needed later; they are

$$
\frac{d\, Q_0(i \cot \alpha)}{d\alpha} = -\, i
$$

$$(21)$$

$$
\frac{d\, Q_2(i \cot \alpha)}{d\alpha} = i\ [3 \operatorname{cosec}^2\alpha\ (1-\alpha \cot \alpha) - 1]
$$

The expressions for Q_m (x) and Q_m $(i \cot \alpha)$ satisfy the differential equations:

$$
(1-x^2)\ \frac{d^2Q_m(x)}{dx^2} - 2x\, \frac{d\, Q_m(x)}{dx} + m(m+1)\, Q_m(x) = 0,
$$

$$(22)$$

$$
\sin^2 \alpha\ \frac{d^2\ Q_m(i \cot \alpha)}{d\alpha^2} - m(m+1)\, Q_m(i \cot \alpha) = 0.
$$

The first of (22) is also satisfied by the Legendre functions or zonal harmonics of the first kinds. For convenience in dealing with real quantities only let us define the quantities

$$
\begin{aligned}
q_0\ (\alpha) &= i\ Q_0(i \cot \alpha) = \alpha, \\
q_1\ (\alpha) &= -\ Q_1(i \cot \alpha) = (1-\alpha \cot \alpha), \qquad\qquad (23)\\
q_2\ (\alpha) &= -\ i\ Q_2(i \cot \alpha) = \frac{1}{2}[(3 \cot^2 \alpha + 1)\ \alpha - 3 \cot \alpha].
\end{aligned}
$$

A table of q_2 and q'_2 is given at the end of this part, for values of α_0 corresponding to values of the earth's flattening expressed as a reciprocal. For the International Ellipsoid of Reference there is also a short table giving $q_2(\alpha)$ and $q'_2(\alpha)$ corresponding to certain elevations above the ellipsoid for points at the equator, the poles, and at geocentric latitude 45°.

Evidently

$$\left.\begin{aligned} \frac{d\, q_2(\alpha)}{d\alpha} &= q_2'(\alpha) = 3 \operatorname{cosec}^2\alpha\; q_1(\alpha) - 1, \\ \frac{d^2\, q_2(\alpha)}{d\alpha^2} &= q_2''(\alpha) = 6 \operatorname{cosec}^2\alpha\; q_2(\alpha)\,. \end{aligned}\right\} \tag{24}$$

In solutions of the type (13) evidently a q may be substituted for a Q.

§2 – The Expression for the Potential.

The potential of the centrifugal acceleration U in rectangular coordinates is given by

$$U = \frac{\omega^2}{2} (x^2 + y^2), \tag{25}$$

ω being the angular velocity of rotation. In our special ellipsoidal coordinates it is

$$U = \frac{\omega^2}{2} c^2 \operatorname{cosec}^2 \alpha \operatorname{Sech}^2\beta, \tag{26}$$

or in terms of Legendre's coefficients

$$U = \frac{\omega^2 c^2 \operatorname{cosec}^2\alpha}{3} [1 - P_2(\tanh \beta)]. \tag{27}$$

The coordinate system is determined by the position of the common foci. Let us take these foci of the ellipsoid of revolution that is to form an equipotential surface for the combined mass-attraction and centrifugal force, and let α_0 be the value of α corresponding to this particular ellipsoid. From equation (2) it is seen that sin α_0 equals the eccentricity of the meridian ellipse, so that for the International Ellipsoid of Reference, for which the ellipticity is 1/297, $\alpha_0 = 4° \; 42' \; 11''.048 = 0.0820840$ radians. Then the problem consists in building up an expression for the potential W = U + V (V being the potential due to mass-attraction), in which U and V are expressed in

terms of α, β, and λ, which shall satisfy the following conditions:

(1) V satisfies (12)

(2) $V = o$ at an infinite distance from the origin,

(3) $W = (U + V)$ is independent of β and λ, when α is put equal to α_o; that is, since W is a function of α_o only, W is constant on the ellipsoid $\alpha = \alpha_o$, which is, therefore, an equipotential surface.

(4) V must approximate to $\frac{kM}{r}$ as r increases, k being the gravitation constant, M the mass and r the radius vector.

Condition (1) is satisfied by any solution formed of special solutions of either type shown by equation (11). The semi-axes of the coordinate ellipsoid being c cosec α and c cot α, it is seen that $\alpha = o$ corresponds to a point at infinity, and that only solutions of the second type under (13) may be used if condition (2) is to be satisfied, for the $P_m^n(i \cot \alpha)$ of the first type increases without limit when α approaches zero, whereas as may be seen from the series (19) by putting i cot α for x, $\frac{d^n Q_m(i \cot \alpha)}{d (i \cot \alpha)^n}$ approaches zero with α. It is clear that since the equipotential surface is to be symmetrical about the axis of rotation, there can be no terms involving the longitude, that is, $n = o$ in all cases, and since the surface is symmetrical about the equatorial plane, there must be no terms that change sign with α; this condition excludes from V terms in which m is odd. By using (27) and noting that P_o (Tanh β) = 1 we therefore can write

$$W = U + V = \frac{\omega^2 c^2 \operatorname{cosec}^2 \alpha}{3} + A_o Q_o(i \cot \alpha)$$
$$+ P_2 (\tanh \beta) \left[A_2 Q_2(i \cot \alpha) - \frac{\omega^2 c^2 \operatorname{cosec}^2 \alpha}{3}\right] + A_4 P_4(\tanh \beta) Q_4(i \cot \alpha) \quad (28)$$
$$+ A_6 P_6 (\tanh \beta) Q_6(i \cot \alpha) + \cdots.$$

The powers of i necessary to make the terms real are supposed to be included in the A's. If, according to condition (3), W is to be independent of β when $\alpha = \alpha_0$ then $o = A_4 = A_6 = A_8 = \cdot \cdot \cdot \cdot$.

and
$$A_2 Q_2(i \cot \alpha_0) - \frac{\omega^2 c^2 \operatorname{cosec}^2 \alpha_0}{3} = 0 \tag{29}$$

which determines A_2. Accordingly (28) becomes, after recombining the separate parts of the expression for the centrifugal force into their original form (26), and substituting for Q_2 in terms of q_2

$$W = A_0 Q_0(i \cot \alpha) + \frac{\omega^2 c^2 \operatorname{cosec}^2 \alpha}{2} \operatorname{Sech}^2 \beta + \frac{\omega^2 c^2 \operatorname{cosec}^2 \alpha_0}{3\, q_2\, \alpha_0} P_2 (\operatorname{Tanh} \beta)\, q_2(\alpha). \tag{30}$$

We can satisfy condition (4) by choosing A_0 suitably; to do this we must ascertain the relation between α and r, when r is large or α small. By squaring and adding equations (1) we get

$$r = c\sqrt{\operatorname{cosec}^2 \alpha \operatorname{Sech}^2 \beta + \cot^2 \alpha \operatorname{Tanh}^2 \beta}\,. \tag{31}$$

But the expansions for cosec α and cot α in power series are $\cot \alpha = \frac{1}{\alpha} - \frac{\alpha}{3} \cdot \cdot \cdot$ and $\operatorname{cosec} \alpha = \frac{1}{\alpha} + \frac{\alpha}{6} \cdot \cdot \cdot \cdot$.

These substituted in (31) give

$$r = c\sqrt{\frac{1}{\alpha^2} - \frac{2}{3} \operatorname{Tanh}^2 \beta + \frac{1}{3} \operatorname{Sech}^2 \beta + \text{terms in } \alpha^2, \text{ etc.}}\ ,$$

from which it appears that $r = \frac{c}{\alpha}$ is a good approximation when α is small or r large.

By equation (17) $Q_2(i \cot \alpha)$ or $q_2(\alpha)$ is of the third order when α is small, α being of the first, so the term in (30) which involves $q_2(\alpha)$ need not be considered in satisfying condition (4). It is thus seen that condition (4) is satisfied by taking $A_0 = \frac{R M i}{c}$ and the expression (28)

for W becomes

$$W = \frac{k\,M\,\alpha}{c} + \frac{\omega^2\,c^2\,\mathrm{cosec}^2\,\alpha}{2}\,\mathrm{Sech}^2\,\beta + \frac{\omega^2\,c^2\,\mathrm{cosec}^2\,\alpha_0}{3\,q_2(\alpha_0)}\,P_2(\mathrm{Tanh}\,\beta)\,q_2(\alpha). \quad (32)$$

This equation satisfies all conditions and is therefore the potential function that has the ellipsoid of revolution $\alpha = \alpha_0$ as an equipotential surface for the combined effects of mass-attraction and centrifugal acceleration.

§3 - The formula for the Intensity of Gravity.

As a preparation for obtaining the gravity formula let us derive the distance between the two ellipsoids characterized by the coordinates α and $\alpha + \Delta\alpha$, $\Delta\alpha$ being infinitesimal. Take a point P on the first ellipsoid whose coordinates are (α, β, λ), or in rectangular coordinates (x, y, z), and a point Q on the second whose coordinates are $(\alpha + \Delta\alpha, \beta, \lambda)$, or in rectangular coordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$. Since the system of coordinates is orthogonal, the infinitesimal line of length $PQ = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ will be perpendicular to both ellipsoids and will be the distance required. Since α alone varies

$$\Delta x = \frac{\partial x}{\partial \alpha}\Delta\alpha + \text{terms of higher order,}$$

$$\Delta y = \frac{\partial y}{\partial \alpha}\Delta\alpha + \text{terms of higher order,}$$

$$\Delta z = \frac{\partial z}{\partial \alpha}\Delta\alpha + \text{terms of higher order.}$$

Or by differentiating the equations (1), squaring, adding and simplifying

$$PQ = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = c\,\mathrm{cosec}\,\alpha\,\sqrt{\cot^2\alpha + \mathrm{Tanh}^2\,\beta}\;\Delta\alpha + \text{terms of}$$

higher order, or since PQ is an element of the normal n,

$$\Delta n = \frac{\partial n}{\partial \alpha}\Delta a = c\,\mathrm{cosec}\,\alpha\,\sqrt{\cot^2\alpha + \mathrm{Tanh}^2\,\beta}\;\Delta a + \text{terms of higher} \quad (33)$$

order. Since α decreases outward from the origin, the above expression

implies that the normal $\underline{n}$ is directed inward.

For brevity let us put

$$h = c \operatorname{cosec} \alpha \sqrt{\cot^2 \alpha + \operatorname{Tanh}^2 \beta}, \tag{34}$$

so that equation (33) is equivalent to

$$\frac{\partial n}{\partial \alpha} = h$$

It will be convenient to have the special value at the ellipsoid $\alpha = \alpha_0$. If we call $\underline{a}$ the semi-axis of the ellipsoid, $a = c \operatorname{cosec} \alpha_0$ and $e = \sin \alpha_0$. Therefore by (8) and (9), equation (32) gives, after a slight reduction

$$\frac{1}{h_0} = \frac{\tan \alpha_0}{a \sqrt{1 + \tan^2 \alpha_0 \sin^2 u}} . \tag{35}$$

The subscript o attached to $\underline{h}$, indicates the value for the ellipsoid $\alpha = \alpha_0$.

If we introduce the geographic latitude Φ instead of the reduced latitude u, we find for h_0 after some transformation by the help of equations (7)

$$h_0 = \frac{a \cot \alpha_0}{\sqrt{1 - \sin^2 \alpha_0 \sin^2 \Phi}} . \tag{36}$$

The value of the intensity of gravity (g) at any point may be found by dividing the difference in potential between the two ellipsoids at an infinitesimal distance from one another by that distance, and dropping the terms of higher order which vanish when the limit is taken.

$$\begin{aligned} g &= \frac{1}{h} \cdot \frac{\partial W}{\partial \alpha} \\ &= \frac{\partial W}{\partial \alpha} \cdot \frac{\sin \alpha}{c \sqrt{\cot^2 \alpha + \operatorname{Tanh}^2 \beta}} \end{aligned} \tag{37}$$

By differentiating (32) to get $\frac{\partial W}{\partial \alpha}$, and using (24) we find

$$g = \left[\frac{kM}{c} - \omega^2 c^2 \operatorname{cosec}^2\alpha \cot\alpha \operatorname{Sech}^2\beta + \frac{\omega^2 c^2 \operatorname{cosec}^2\alpha_0}{3q_2(\alpha_0)} q_2'(\alpha) P_2(\operatorname{Tanh}\beta)\right] \times \left[\frac{\sin\alpha}{c\sqrt{\cot^2\alpha + \operatorname{Tanh}^2\beta}}\right] \tag{38}$$

This is the expression for the intensity of gravity at any point (α, β, λ) outside the ellipsoid $\alpha = \alpha_0$. The case when the point is on this ellipsoid is included as a limiting case, and for this case (38) can be simplified.

Let us put $\alpha = \alpha_0$ and introduce the abbreviation F defined by

$$F = \frac{q_2'(\alpha_0) \sin\alpha_0}{3\, q_2(\alpha_0)} \tag{39}$$

In the first expression F is indeterminate in form when $\alpha = 0$, and this expression is therefore inconvenient for purposes of computation when α_0 is small, as in the case of an ellipsoid approximating the figure of the earth. In this case it is convenient to use the series (19) for $Q_2(i\cot\alpha)$, which gives

$$q_2(\alpha) = -i\,Q_2(i\cot\alpha) = \frac{2}{15}[\tan^3\alpha - \frac{6}{7}\tan^5\alpha + \frac{5}{7}\tan^7\alpha - \frac{20}{33}\tan^9\alpha \cdots]$$

$$= \frac{2}{3.5}\tan^3\alpha - \frac{4}{5.7}\tan^5\alpha + \frac{6}{7.9}\tan^7\alpha - \frac{8}{9.11}\tan^9\alpha \cdots$$

and

$$q_2'(\alpha) = \frac{2\sec^2\alpha}{15}[3\tan^2\alpha - \frac{30}{7}\tan^4\alpha + 5\tan^6\alpha - \frac{60}{11}\tan^8\alpha \cdots],$$

$$= \sec^2\alpha[\frac{2\tan^2\alpha}{5} - \frac{4\tan^4\alpha}{7} + \frac{6\tan^6\alpha}{9} - \frac{8\tan^8\alpha}{11} \cdots]$$

so that

$$F = \frac{\sec\alpha_0[1 - \frac{10}{7}\tan^2\alpha_0 + \frac{5}{3}\tan^4\alpha_0 - \frac{20}{11}\tan^6\alpha_0 + \cdots]}{1 - \frac{6}{7}\tan^2\alpha_0 + \frac{5}{7}\tan^4\alpha_0 - \frac{20}{33}\tan^6\alpha_0 + \cdots}, \tag{40}$$

or approximately by division

$$F = \sec\alpha_0[1 - \frac{4}{7}\tan^2\alpha_0 + \frac{68}{147}\tan^4\alpha_0 - \frac{4612}{11319}\tan^6\alpha_0 + \cdots]. \tag{41}$$

Putting $\alpha = \alpha_0$ in (38), introducing F, substituting for β in terms of u and simplifying gives for g_0, the value of gravity on the ellipsoid,

$$g_0 = \left\{\frac{k\,M}{a^2} - \omega^2 a \cos\alpha_0 \cos^2 u + \omega^2 a\, F\, P_2(\sin u)\right\} \frac{1}{\sqrt{1 - \sin^2\alpha_0 \cos^2 u}}. \quad (42)$$

Instead of the constants M and ω let us express the result in terms of g_e, the value of gravity at the equator, and m the ratio of the centrifugal acceleration at the equator to gravity there, or $m = \omega^2\, a/g_e$.

By putting u = o, and $\omega^2\, a = m\, g_e$ in (42) and solving for $k\,M/a^2$ we find

$$\frac{k\,M}{a^2} = g_e\,[\cos\alpha_0 + m\,(\tfrac{1}{2}\,F + \cos\alpha_0)]. \quad (43)$$

By using this (42) reduces to

$$g_0 = \frac{g_e}{\sqrt{1 + \tan^2\alpha_0 \sin^2 u}}\,[1 + m\,(\tfrac{3}{2}\,F \sec\alpha_0 + 1) \sin^2 u]. \quad (44)$$

This is an exact expression for gravity on the surface of the ellipsoid at a point, the reduced latitude of which is u, $\sin\alpha_0$ being equal to e, the eccentricity of the ellipsoid and F being given by (39), (40) or (41). This can be expressed directly in terms of the geographic latitude Φ. From (7)

$$\sin u = \frac{\sin\Phi \cos\alpha_0}{\sqrt{1 - \sin^2\alpha_0 \sin^2\Phi}}$$

and after substituting this in (44) and reducing we get

$$g_0 = g_e\sqrt{1 - \sin^2\alpha_0 \sin^2\Phi}\,\left\{1 + m\cos\alpha_0\,\frac{(\tfrac{3}{2}\,F + \cos\alpha_0)\sin^2\Phi}{1 - \sin^2\alpha_0 \sin^2\Phi}\right\}. \quad (45)$$

This expression is exact for an ellipsoid of any eccentricity.

On the assumptions stated, the expression for g_o is rigorous. Moreover it is in finite terms, for the quantity $\cos \alpha_o(\frac{3}{2} F + \cos \alpha_o)$ is in any given case merely a numerical constant, although the way in which this quantity involves α_o is somewhat complicated.

§4 - Various Deductions from the Formula for Gravity.

Clairaut's Equation.

By putting $\Phi = 90°$ in (45) or $u = 90°$ in (44) we get for g_p, the value of gravity at the pole

$$g_p = g_e \cos \alpha_o \left\{1 + m(\tfrac{3}{2} F \sec \alpha_o + 1)\right\}, \tag{46}$$

and by forming the expression for the quantity

$$\beta = \frac{g_p - g_e}{g_e}$$

we find

$$\beta = \frac{g_p - g_e}{g_e} = (\tfrac{3}{2} F + \cos \alpha_o)\, m - (1 - \cos \alpha_o) \tag{47}$$

Let us introduce the flattening f, which is connected with α_o by the relation

$$e^2 = \sin^2 \alpha_o = 2f - f^2. \tag{48}$$

Let us consider m, e^2 and f as small quantities of the same order of magnitude; as a matter of fact m, being equal to about 1/288, is very nearly equal to f. If in (47) we consider only quantities of the first order, we get by using (41)

$$\beta = \frac{g_p - g_e}{g_e} = \frac{5}{2} m - f. \tag{49}$$

This is the familiar form of Clairaut's equation and, though accurate to the terms of the first order only, is sufficient for many purposes. It is desirable, however, to include terms of higher order.

By putting in (41) $\sec \alpha_0 = \sqrt{1 + \tan^2 \alpha_0} = 1 + \frac{1}{2} \tan^2 \alpha_0 - \frac{1}{8} \tan^4 \alpha_0 \cdots$ multiplying out this series and the other series in (41) and substituting $\tan^2 \alpha_0 = \frac{2f - f^2}{(1-f)^2} = 2f + 3f^2 + 4f^3 \cdots$,

also noting that $1 - \cos \alpha_0 = f$, we find after reduction

$$\beta = \frac{g_p - g_e}{g_e} = \frac{5}{2} m \left(1 - \frac{17}{35} f - \frac{1}{245} f^2 \cdots\right) - f = \frac{5}{2} m \chi(f) - f \qquad (50)$$

the function $\chi(f)$ being defined by the series in parentheses.

This is Clairaut's equation extended to terms of even higher order than are needed in practice. 3/

Equation (50) applies, of course, only when the level surface is an exact ellipsoid of revolution.

Relation Between the Newtonian Constant k and the Mean Density of the Earth.

The mean density ρ_m is defined by the condition that the mass of a homogeneous ellipsoid of the adopted dimensions shall have the same mass, that is, that

$$\frac{4}{3} \pi a^2 b \rho_m = \frac{4}{3} \pi a^3 \cos \alpha_0 \rho_m = M \qquad (51)$$

Substituting this value of M in (43) and expressing F and α_0 in terms of the flattening f we get

$$k \rho_m = \frac{3 g_e}{4 \pi a} \left[1 + \frac{3}{2} m \left(1 + \frac{2}{7} f + \frac{125}{441} f^2 \cdots\right)\right]. \qquad (52)$$

3/ An equivalent form is given without proof by Cassinis in the Bulletin Géodésique, No. 26, April-May-June, 1930, p. 40. Cassinis gives also an additional term for $\chi(f)$ following $-\frac{1}{245} f^2$, namely $-\frac{13}{18865} f^3$. The correctness of this additional term has been verified by the writer.

The Mean Value of Gravity

It is understood that the mean is to be taken with regard to area. The value of mean gravity, g_m, is therefore given by

$$g_m = \frac{\int g_o dS}{\int dS}$$

Where dS is an element of area of the ellipsoid and the integration covers the entire area.

The reduced latitude u is convenient in evaluating these integrals. In this case we have $dS = 2\pi x ds$ where ds is an element of meridional arc. From $x = a \cos u$, $y = b \sin u$

we find

$$dS = 2\pi ab \cos u \sqrt{1 + \tan^2 \alpha_o \sin^2 u}\, du.$$

As is well known, this may be evaluated in closed form in terms of the elementary functions but the expression obtained is inconvenient to calculate when $\tan \alpha_o$ is small. It is, therefore, simplest for the purposes of practical computation to expand $\sqrt{1 + \tan^2 \alpha_o \sin^2 u}$ in powers of sin u and integrate term by term.

In this way we find

$$S = \int_{u=-\frac{\pi}{2}}^{u=\frac{\pi}{2}} dS = 4\pi ab \left[1 + \frac{1}{6}\tan^2 \alpha_o - \frac{1}{40}\tan^4 \alpha_o + \frac{1}{112}\tan^6 \alpha_o \cdots\right]. \quad (53)$$

In calculating $\int g_o\, dS$ it is convenient again to expand $\sqrt{1 + \tan^2 \alpha_o \sin^2 u}$ in powers of sin u. In this way by using equation (44) for g_o we find

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_o dS = 2\pi abg_e \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+m(\tfrac{3}{2} F \sec \alpha_o + 1) \sin^2 u] \cos u \, du$$

$$= 4\pi abg_e \, [1 + \frac{m}{3} (\tfrac{3}{2} F \sec \alpha_o + 1)]$$

$$= 4\pi abg_e \, [1 + \frac{5}{6} m (1 + \frac{9}{35} \tan^2\alpha_o - \frac{16}{245} \tan^4\alpha_o \cdots)].$$

Therefore

$$g_m = \frac{\int g_o dS}{S} = \frac{g_e[1 + \frac{5}{6} m(1 + \frac{9}{35} \tan^2\alpha_o - \frac{16}{245} \tan^4\alpha_o \cdots]}{1 + \frac{1}{6} \tan^2\alpha_o - \frac{1}{40}\tan^4\alpha_o + \frac{1}{112} \tan^6\alpha_o \cdots}$$

$$\begin{aligned} = g_e \, [1 - \frac{1}{6} \tan^2\alpha_o + \frac{19}{360} \tan^4\alpha_o - \frac{331}{15120} \tan^6\alpha_o \cdots \\ + \frac{5}{6} m(1 + \frac{19}{210}\tan^2\alpha_o - \frac{977}{17640} \tan^4\alpha_o \cdots)] \end{aligned} \quad (54)$$

$$\begin{aligned} = g_e \, [1 - \frac{1}{3} f - \frac{13}{45} f^2 - \frac{197}{945} f^3 \cdots \\ + \frac{5}{6} m(1 + \frac{19}{105} f + \frac{22}{441} f^2 \cdots)]. \end{aligned} \quad (55)$$

The value g_m of mean gravity is of interest because in practice it is usually better determined than gravity at the equator. Formulas based on rather different values of gravity at the equator and on rather different flattenings often give nearly the same value for mean gravity.

§5- Other Forms of the Expression for Theoretical Gravity.

The quantity $\frac{3}{2}$ F cos α_o + $\cos^2 \alpha_o$ depends merely on the flattening of the ellipsoid. For small flattenings its value is approximately $\frac{5}{2}$. For brevity let us denote this quantity $\frac{3}{2}$ F cos α_o + $\cos^2 \alpha$ by C. The value of C is obtained from (40) by multiplying by cos α_o and dividing the denominator into the numerator. We find in terms of tan α.

$$F \cos \alpha_o = 1 - \frac{4}{7} \tan^2 \alpha_o + \frac{68}{147} \tan^4 \alpha_o - \frac{4612}{11319} \tan^6 \alpha_o \cdots \quad (56)$$

Since

$$\tan^2 \alpha_o = \frac{\sin^2 \alpha_o}{1-\sin^2 \alpha_o} = \frac{e^2}{1-e^2} = e^2 + e^4 + e^6 \cdots \quad (57)$$

We find by substitution

$$F \cos \alpha_o = 1 - \frac{4}{7} e^2 - \frac{16}{147} e^4 - \frac{608}{11319} e^6 \cdots \quad (58)$$

We have in terms of the flattening f by means of the relation

$$e^2 = 2f - f^2, \quad (59)$$

$$F \cos \alpha_o = 1 - \frac{8}{7} f + \frac{20}{147} f^2 + \frac{64}{11319} f^3 \cdots \quad (60)$$

The forms (58) and (60) are more convergent than (56).

In terms of e^2 and f we have

$$C = \frac{5}{2} - \frac{13}{7} e^2 - \frac{8}{49} e^4 - \frac{304}{3773} e^6 \cdots \quad (61)$$

$$= \frac{5}{2} - \frac{26}{7} f + \frac{59}{49} f^2 + \frac{32}{3773} f^3 \cdots \quad (62)$$

Returning to (45) we write it

$$g_o = g_e \left\{ \sqrt{1-e^2 \sin^2 \Phi} + \frac{m\, C \sin^2 \Phi}{\sqrt{1-e^2 \sin^2 \Phi}} \right\}. \quad (63)$$

This form suggests an expansion in powers of $\sin^2\Phi$; we, therefore, assume

$$g_0 = g_e[1 + C_2 \sin^2\Phi + C_4 \sin^4\Phi + C_6 \sin^6\Phi \cdot \cdot \cdot]. \tag{64}$$

By expanding the radicals in (63) and noting that $\underline{m}$ is a quantity of the same order as e^2 we find

$$\left.\begin{aligned} C_2 &= \frac{1}{2}(2mC - e^2) \\ C_4 &= \frac{e^2}{8}(4mC - e^2) \\ C_6 &= \frac{e^4}{16}(6mC - e^2) \\ C_8 &= \frac{5e^6}{128}(8mC - e^2) \end{aligned}\right\} \tag{65}$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$C_{2k} = \frac{1\cdot 1\cdot 3\cdot 5 \cdot \cdot \cdot (2k-3)}{2\cdot 4\cdot 6\cdot 8 \cdot \cdot \cdot \quad 2k} e^{2k-2}[2kmC - e^2]. \tag{66}$$

The numerical coefficient of C_{2k} is that of y^{2k} in the expansion of $\sqrt{1 - y^2}$. The numerical values of the C's will be given in §6.

For actual computation of tables the expansion in multiple angles is probably as convenient as any. If we put

$$g_0 = \frac{1}{2} B_0 + B_2 \cos 2\Phi + B_4 \cos 4\Phi + B_6 \cos 6\Phi \cdot \cdot \cdot \tag{67}$$

and use the expressions for even powers of the sine in terms of the cosines of multiple angles, namely,

$$\sin^2\Phi = \frac{1}{2} - \frac{1}{2} \cos 2\Phi$$

$$\sin^4\Phi = \frac{3}{8} - \frac{1}{2} \cos 2\Phi + \frac{1}{8} \cos 4\Phi$$

$$\sin^6\Phi = \frac{5}{16} - \frac{15}{32} \cos 2\Phi + \frac{3}{16} \cos 4\Phi - \frac{1}{32} \cos 6\Phi$$

$$\sin^8\Phi = \frac{35}{128} - \frac{7}{16} \cos 2\Phi + \frac{7}{32} \cos 4\Phi - \frac{1}{16} \cos 6\Phi + \frac{1}{128} \cos 8\Phi$$

we get by collecting coefficients

$$\begin{aligned}
\frac{1}{2} B_0 &= g_e\left(1 + \frac{1}{2} C_2 + \frac{3}{8} C_4 + \frac{5}{16} C_6 + \frac{35}{128} C_8 \cdots\right) \\
B_2 &= -g_e\left(\frac{1}{2} C_2 + \frac{1}{2} C_4 + \frac{15}{32} C_6 + \frac{7}{16} C_8 \cdots\right) \\
B_4 &= g_e\left(\frac{1}{8} C_4 + \frac{3}{16} C_6 + \frac{7}{32} C_8 \cdots\right) \\
B_6 &= -g_e\left(\frac{1}{32} C_6 + \frac{1}{16} C_8 \cdots\right) \\
B_8 &= g_e\left(\frac{1}{128} C_8 \cdots\right)
\end{aligned} \tag{68}$$

The coefficients of the expansions of $(1 - e^2 \sin^2\Phi)^{\pm\frac{1}{2}}$ may be expressed as hypergeometric series in terms of e^2 or $n = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}} = \tan^2 \frac{\alpha_0}{2}$, which latter series are even more rapidly convergent than those in e^2. The B's of equations (68) may also be expressed as elliptic integrals. The functions q_1 and q_2 are also expressible as hypergeometric series, but for the purpose in hand the elementary treatment here given seems adequate.

Cassinis 4/ puts the formula for gravity in a form resembling that previously in use, namely

$$\begin{aligned}
g_0 = g_e(1 + \beta \sin^2\Phi - \beta_1 \sin^2 2\Phi - \beta_2 \sin^2\Phi \sin^2 2\Phi \\
-\beta_3 \sin^4\Phi \sin^2 2\Phi \cdots),
\end{aligned} \tag{69}$$

a form in which all terms after the term in β contain the factor $\sin^2 2\Phi$ and thus vanish both at equator and pole. It will be seen that this β is the same as the β of equations (47) and (50). Helmert's formula of 1901 stops with the term in β_1 5/. As a matter of fact in

4/ G. Cassinis. Bulletin Géodésique, No. 26, April-May-June, 1930, p. 40

5/ Ber. d. Kön. Pr. Akad. d. Wiss., XIV 1901, S.328.

the present instance the succeeding terms are very small. The values of the β's can be determined in terms of the C's previously used by means of the relation (68).

By substituting this in equation (69), collecting terms and comparing coefficients we find

$$\begin{aligned} \beta &= C_2 + C_4 + C_6 + C_8 \cdots . \\ \beta - 4\beta_1 &= C_2 , \end{aligned} \tag{70}$$

or

$$\begin{aligned} \beta_1 &= \frac{1}{4}(C_4 + C_6 + C_8 \cdots). \\ 4(\beta_1 - \beta_2) &= C_4 , \end{aligned} \tag{71}$$

or

$$\begin{aligned} \beta_2 &= \frac{1}{4}(C_6 + C_8 \cdots). \\ 4(\beta_2 - \beta_3) &= C_6 , \end{aligned} \tag{72}$$

or

$$\beta_3 = \frac{1}{4}(C_8 \cdots) \tag{73}$$

Another form given by Cassinis (l.a.c)

By reducing to a common denominator within the braces and using, as before, the abbreviation C for $\frac{3}{2} F \cos \alpha_o + \cos^2 \alpha_o$, (45) may be written

$$g_o = \frac{g_e[1 + (mC - e^2)\sin^2\Phi]}{\sqrt{1 - e^2 \sin^2\Phi}} . \tag{74}$$

It can be shown that

$$mC - e^2 = \beta - f - f\beta, \tag{75}$$

so that (74) may be written in Cassinis's form

$$g_o = \frac{g_e[1 + (\beta - f - f\beta)\sin^2\Phi]}{\sqrt{1 - e^2\sin^2\Phi}} = \frac{g_e[1 + (\beta - f - f\beta)\sin^2\Phi]}{\sqrt{1 - f(2 - f)\sin^2\Phi}} \tag{76}$$

The quantity $W = 1 - e^2\sin^2\Phi$ is given in the tables for the International Ellipsoid of Reference 6/ from $\Phi = 0°$ to $\Phi = 45°$ and of the quantity $V = \frac{b}{a} W = \sqrt{1-e^2}\, W$ from $\Phi = 45°$ to $\Phi = 90°$. But if these tables are to be used for the systematic calculation of g_0, it is simpler to use a transformation involving the "great normal" 7/ N, a transformation that avoids the discontinuity in tabulation at $\Phi = 45°$ and also the necessity of passing from W^2 to W. By using the relation between N and V or W we may write

$$\begin{aligned} g_0 &= g_e \frac{N}{a}[1 + (\beta - f - f\beta)\sin^2\Phi] \\ &= g_e \frac{N}{a}[1 + (mC - e^2)\sin^2\Phi]. \end{aligned} \tag{77}$$

The quantity g_e/a is constant and the logarithm of N may be taken directly from the tables.

A symmetrical formula involving the shape of the ellipsoid and the values of gravity at Equator and pole, or g_e and g_p, respectively, has been given by Somigliana 8/. Let j denote the ratio of the semi-major to the semi-minor axis or $a/b = 1/\sqrt{1-e^2}$ then Somigliana finds

$$g_0^2(j^2\cos^2\Phi + \sin^2\Phi) = (j\, g_e\cos^2\Phi + g_p\sin^2\Phi)^2. \tag{78}$$

A similar formula may be written in terms of the reduced latitude u. It reads

$$g_0^2(j^2\sin^2 u + \cos^2 u) = (g_e\cos^2 u + j\, g_p\sin^2 u)^2. \tag{79}$$

6/ Tables de l'Ellipsoïde de Référence international --- calculées sous la direction du Général G. Perrier --- par E. Hasse, Paris, 1928.

7/ This quantity is also the radius of curvature in the prime vertical.

8/ C. Somigliana. Atti della Reale Accademia Nazionale dei Lincei, Anno CCCXXIV (1927), Series 6, Vol. 5, p. 319.

The proof of the first may be taken from Somigliana's paper, or the proofs of both may readily be derived by the reader.

§6 - Numerical Values

For the International Ellipsoid the value of the semi-major axis a is 6 378 388 meters and the value of the flattening f is 1/297. From these we find

$$\log a = 6.804\ 71093$$
$$\log f = 7.527\ 24355 - 10$$

The period T of rotation of the earth from star to star is 86164.09890 mean solar seconds. This period is nearly the same as the "sidereal" day so-called expressed in mean solar seconds but should be used rather than the so-called sidereal day in computing the centrifugal acceleration. The so-called sidereal day is really the equinoctial day or interval between successive transits of the vernal equinox. A more appropriate name for the so-called sidereal day would be the equinoctial day. From this value we find

$$\omega = 2\pi/T = 0.000\ 07292\ 11515$$
$$\log \omega = 5.8628\ 53518 - 10$$

The Association of Geodesy on the recommendation of its Gravity Committee adopted as the value of g_e, or gravity at the equator at sea level, 978.049 cm/sec^2. This value is based on the work of Heiskanen 9/. With these values of a, ω, and g_e we find for $m = \omega^2\ a/g_e$

$$\log m = 7.5400\ 57356 - 10$$

9/ W. Heiskanen. Ist die Erde ein dreiachsiges Ellipsoid? Gerlands Beiträge zur Geophysik, Vol. XIX. 1928, p. 356.

With these values as a basis we find for the various quantities for which formulas have been given

$$C = 2.487\ 50763\ 8$$
$$C_2 = 0.005\ 26490\ 98$$
$$C_4 = 0.000\ 02334\ 64$$
$$C_6 = 0.000\ 00012\ 72$$
$$C_8 = 0.000\ 00000\ 07$$

$$\beta = 0.005\ 28838\ 41$$
$$\beta_1 = 0.000\ 00586\ 86$$
$$\beta_2 = 0.000\ 00003\ 20$$

$$\chi(f) = 0.998\ 36455\ 21$$

$$\frac{1}{2} B_0 = 980.632\ 272 \text{ cm/sec}^2$$
$$B_2 = -\ 2.586\ 145$$
$$B_4 = +\ 0.002\ 878$$
$$B_6 = -\ 0.000\ 004$$
$$\log(\beta - f - f\beta) = 7.279\ 5699 - 10$$
$$\log g_e/a = 6.185\ 6496\ 8 - 10$$

We have, therefore, the following working formulas for computing the acceleration of gravity at sea level in latitude Φ, the unit being the gal (cm/sec^2)

$$g_0 = 978.049[1 + 0.005\ 26491 \sin^2\Phi + 0.000\ 02335 \sin^4\Phi + 0.000\ 00013 \sin^6\Phi]. \quad \text{(A)}$$

$$g_0 = 978.049[1 + 0.005\ 28838 \sin^2\Phi - 0.000\ 00587 \sin^2 2\Phi + 0.000\ 00003 \sin^2\Phi \sin^2 2\Phi]. \quad \text{(B)}$$

$$g_0 = 980.632\ 272 - 2.586145 \cos 2\Phi + 0.002\ 878 \cos^4\Phi - 0.000\ 004 \cos^6\Phi. \quad \text{(C)}$$

$$g_0 = [6.185\ 64968 - 10]\, N \left\{1 + [7.2795699 - 10]\sin^2\Phi\right\}. \quad \text{(D)}$$

In conformity with a usual convention among astronomers, the square brackets in the formula (D) indicate that the number within is the common logarithm of the coefficient concerned.

§7 - The q-functions, Tables and Computation

Expression (32) for the potential and expression (38) for gravity at any point in space external to the ellipsoid—and including the surface of the ellipsoid itself—are exact. At points sufficiently far from the surface of the ellipsoid, the concept of a field of force representing the combined effects of Newtonian attraction and centrifugal force becomes meaningless, but in practice the points likely to be considered are far within the limit of applicability.

These equations, (32) and (38), were derived with the aid of curvilinear coordinates based on three families of mutually orthogonal surfaces, one family being a set of ellipsoids of rotation. But of all these ellipsoids the only one that is of special interest to us is the ellipsoid $\alpha = \alpha_0$, where $\sin^2\alpha_0 = e^2$ = square of eccentricity of meridian ellipse. The condition $\alpha < \alpha_0$ corresponds to ellipsoids of the same family outside the ellipsoid of reference for which $\alpha = \alpha_0$. Unfortunately these outer ellipsoids do not give us much help in formulating our ideas about the level surfaces outside of $\alpha = \alpha_0$. The ellipsoids of increasing axes, corresponding to decreasing values of α, become less and less flattened as we recede from the center. On the contrary, the actual level surfaces become more and more flattened under the same conditions.[10]

[10] This is due to the inclusion of the centrifugal force. The level surfaces of the Newtonian attraction alone would become more and more nearly spherical.

Moreover, the level surfaces are not exact ellipsoids but spheroids depressed in middle latitudes below the ellipsoids having the same equatorial and polar axes.

It would seem, therefore, that expressions (32) and (38) would seldom be used, so that the values of $q_2(\alpha)$ and $q_2'(\alpha)$, where $\alpha = \alpha_0$, could be computed as occasion might require. Thus, it would also seem that elaborate tables of these functions would be of little use.

But in order to give an idea of the general trend of these functions, two tables have been prepared. Table I gives the values of $q_2(\alpha_0)$ and $q_2'(\alpha_0)$ for values of α_0 corresponding to ellipticities (flattenings) denoted by

$$f = 2 \sin^2 \frac{\alpha_0}{2} = \frac{1}{296.0}, \frac{1}{296.1}, ----- \frac{1}{298.5}.$$

This range covers all modern ellipsoids of reference likely to be of interest. For obvious reasons, figures for three special values outside this range are given, namely for the ellipsoids of Clarke (1880), Clarke (1866), and Bessel.

Table I applies to the ellipsoid of reference only, as is indicated by the subscript zero.

Table II is for the International Ellipsoid only (flattening equals 1/297 exactly) but applies to a region above (outside) this ellipsoid. The argument is the altitude of a point above the equator. The table gives the values of α, $q_2(\alpha)$, and $q_2'(\alpha)$ for radii exceeding the radii of the earth by 10, 20, 30, • • • 100 kilometers and at three points, the equator, the pole, and geocentric latitude 45°. At the equator and pole the excess radii, 10, 20, 30 • • • kilometers, are

altitudes, but clearly at latitude 45° the excess radius is not exactly an altitude, whatever definition we may give to the concept altitude.

If Δr is the excess radius, the difference between Δr and altitude defined in any reasonable way is of the order of $\frac{f^2}{2}\Delta r$ at latitude 45°, the latitude at which ambiguities in the definition of altitude produce the largest discrepancies. Possible definitions of altitude will be considered in Part III.

The second of equations (3) is readily transformed into

$$\tan^2 \alpha = \frac{2c^2}{\sqrt{(r^2 - c^2)^2 + 4c^2z^2} + r^2 - c^2} = \frac{\sqrt{(r^2 - c^2)^2 + 4c^2z^2} - (r^2 - c^2)}{2z^2} \tag{3a}$$

For the equator we may put

$$x = r, \quad y = z = 0,$$

and for the pole

$$z = r, \quad x = y = 0,$$

whence we find:

at the equator

$$\sin^2 \alpha = \frac{c^2}{r^2}, \qquad \tan^2 \alpha = \frac{c^2}{r^2 - c^2}, \tag{80}$$

at the pole

$$\sin^2 \alpha = \frac{c^2}{r^2 + c^2}, \qquad \tan^2 \alpha = \frac{c^2}{r^2}. \tag{81}$$

With the usual notation for the semi-axes of the ellipsoid of reference, $c^2 = a^2e^2$. Equations (80) and (81) give for the surface of the ellipsoid at both equator and pole

$$\sin^2\alpha_0 = e^2 = \frac{a^2 - b^2}{a^2},$$

$$\tan^2\alpha_0 = \epsilon^2 = \frac{a^2 - b^2}{b^2},$$

which are merely our adopted definitions, as they should be.

To determine α for a point outside the ellipsoid of reference, it may often be convenient to expand the radicals in (3) or (3a) in series after taking $r^2 + c^2$ or $r^2 - c^2$ outside the radical. Since all meridian ellipses are alike, we put

$$y = 0 ,$$
$$x = r \cos \Psi ,$$
$$z = r \sin \Psi ,$$

where Ψ is the geocentric latitude in the expression for $\sin^2\alpha$.

For brevity call

$$\sin^2\alpha_p = \frac{c^2}{r^2 + c^2} \qquad (82)$$

Then from (3) we find

$$\sin^2\alpha = \sin^2\alpha_p + \sin^4\alpha_p \cos^2\alpha_p \cos^2\Psi + 2 \sin^6\alpha_p \cos^4\alpha_p \cos^4\Psi + 5 \sin^8\alpha_p \cos^6\alpha_p \cos^6\Psi + 14 \sin^{10}\alpha_p \cos^8\alpha_p \cos^8\Psi + \cdots . \qquad (83)$$

From (3a) we find, putting

$$\tan^2\alpha_e = \frac{c^2}{r^2 - c^2}, \qquad (84)$$

$$\tan^2\alpha = \tan^2\alpha_e - \tan^4\alpha_e \sec^2\alpha_e \sin^2\Psi + 2 \tan^6\alpha_e \sec^4\alpha_e \sin^4\Psi - 5 \tan^8\alpha_e \sec^6\alpha_e \sin^6\Psi + 14 \tan^{10}\alpha_e \sec^8\alpha_e \sin^8\Psi + \cdots . \qquad (85)$$

The coefficients 2, ± 5, 14, etc. are those of the expansion of

$$\frac{1}{2}(1 \pm 4x)^{\frac{1}{2}}.$$

The subscripts p and e, suggesting pole and equator, do not mean that we are actually computing for the pole or equator, but merely that the notation is suggested by the similarity in form to $\sin^2\alpha$ in (81) and $\tan^2\alpha$ in (80).

Series Expansions of q_2 and q_2'

From

$$q_2(\alpha) = \frac{1}{2}\left[(3\cot^2\alpha + 1)\,\alpha - 3\cot\alpha\right]$$

we find

$$\frac{dq_2(\alpha)}{d\alpha} = q_2' = 3\cot^2\alpha + 2 - 3\,\alpha\cot\alpha\,\mathrm{cosec}^2\alpha.$$

These expressions are very troublesome to compute when α is small. We use by preference the expansions in series with

$$\tan\alpha = \epsilon,$$

$$q_2(\alpha) = \frac{2}{3.5}\epsilon^3 - \frac{4}{5.7}\epsilon^5 + \frac{6\epsilon^7}{7.9} - \cdots \pm \frac{2n\epsilon^{2n+1}}{(2n+1)(2n+3)} + \cdots,$$

$$q_2'(\alpha) = 6\left(\frac{\epsilon^2}{3.5} - \frac{\epsilon^4}{5.7} + \frac{\epsilon^6}{7.9} - \cdots \pm \frac{\epsilon^{2n}}{(2n+1)(2n+3)} + \cdots\right).$$

If we take out the first term as a factor, we have

$$q_2(\alpha) = \frac{2}{15}\epsilon^3\left[1 - \frac{6}{7}\epsilon^2 + \frac{5}{7}\epsilon^4 - \frac{20}{33}\epsilon^6 + \cdots\right], \tag{86}$$

$$q_2'(\alpha) = \frac{2}{5}\epsilon^2\left[1 - \frac{3.5}{5.7}\epsilon^2 + \frac{3.5}{7.9}\epsilon^4 - \frac{3.5}{9.11}\epsilon^6 + \cdots\right]. \tag{87}$$

The quantities in square brackets may be expressed as hypergeometric series $F(\alpha, \beta; \gamma; x)$ where

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\, \beta(\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} x^2$$

$$+ \frac{\alpha(\alpha+1)\,(\alpha+2) \cdot \beta(\beta+1)\, \beta+4}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot (\gamma+1)\,(\gamma+2)} x^3 + \cdot \cdot \cdot \cdot$$

Then

$$q_2(\alpha) = \frac{2}{15} \epsilon^3 \, F(\tfrac{3}{2}, 2; \tfrac{7}{2}; -\epsilon^2) \tag{88}$$

$$q_2'(\alpha) = \frac{2}{5} \epsilon^2 \, F(1, \tfrac{3}{2}; \tfrac{7}{2}; -\epsilon^2) \tag{89}$$

Evidently α and β may be interchanged at pleasure.

The introduction of the hypergeometric function enables us to derive for q_2 and q_2' other series that may be useful in checking numerical computation. The new series involve quantities familiar to geodesists. In terms of the semi-axes $\underline{a}$ and $\underline{b}$ of the meridian ellipse, ϵ is evidently the second eccentricity, or

$$\epsilon^2 = \frac{a^2 - b^2}{b^2} = \tan^2 \alpha. \tag{90}$$

The ordinary eccentricity e is given by

$$e^2 = \frac{a^2 - b^2}{a^2} = \sin^2 \alpha \quad . \tag{91}$$

The flattening (ellipticity) f is given by

$$f = \frac{a - b}{a} = 2 \sin^2 \frac{\alpha}{2} \, . \tag{92}$$

There is still another quantity n, which seems to have no special name; it is given by

$$n = \frac{a - b}{a + b} = \tan^2 \frac{\alpha}{2} . \tag{93}$$

A general transformation (A) given in elementary treatments of the hypergeometric function is

$$F(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right) \tag{A}$$

Two rather special transformations are given by Kummer [11].

$$F(\alpha, \alpha + \tfrac{1}{2}; \gamma; x) = \left(\frac{1 + \sqrt{1-x}}{2}\right)^{-2\alpha} F\left(2\alpha, 2\alpha-\gamma+1; \gamma, \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right) \tag{B}$$

$$F(\alpha, \alpha + \tfrac{1}{2}; \gamma; x) = (1 - x)^{-\alpha} F\left(2\alpha, 2\gamma-2\alpha-1; \gamma; \frac{\sqrt{1-x} - 1}{2\sqrt{1-x}}\right) \tag{C}$$

Transforms (A), (B), and (C) give us respectively power series in e^2, n, and $\frac{1}{2}f$ with numerical coefficients as follows:

$$\begin{aligned} q_2(\alpha) &= \frac{2}{15} e^2 \left[1 + \frac{9}{14} e^2 + \frac{25}{56} e^4 + \frac{175}{528} e^6 + \frac{4725}{18304} e^8 + \cdots \right], \\ &= \frac{16}{15} n^{\frac{3}{2}} \left[1 - \frac{3}{7} n + \frac{2}{7} n^2 - \frac{50}{231} n^3 + \frac{25}{143} n^4 - \cdots \right], \\ &= \frac{2}{15} e^3 \left[1 + \frac{2}{7} f + \frac{8}{7} f^2 + \frac{200}{231} f^3 + \frac{600}{1001} f^4 + \cdots \right]. \end{aligned} \tag{94}$$

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[11] E. E. Kummer. "Uber die hypergeometrische Reihe

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\, \beta(\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} x^2 + \frac{\alpha(\alpha+1)\,(\alpha+2)\, \beta(\beta+1)\,(\beta+2)}{1 \cdot 2 \cdot 3\, \gamma\, (\gamma+1)\,(\gamma+2)} x^3 + \cdots"$$

(Crelle's) Journal fur die reine und angewandte Mathematik, 1836. Vol 15, 39-83. Kummer gives a great number of transformations. These used here are numbers 42 and 47 on page 77.

$$q_2' = \frac{2}{5} e^2 \left[1 + \frac{4}{7} e^2 + \frac{8}{21} e^4 + \frac{64}{231} e^6 + \frac{640}{3003} e^8 + \cdots \right],$$

$$= \frac{8}{5} n \left[1 + \frac{2}{7} n - \frac{n^2}{21} + \frac{4}{231} n^3 - \frac{25}{3003} n^4 + \cdots \right], \tag{95}$$

$$= \frac{2}{5} e^2 \left[1 + \frac{8}{7} f + \frac{20}{21} f^2 + \frac{160}{231} f^3 + \frac{200}{429} f^4 \cdots \right].$$

The series in n are the most advantageous for computation, since the signs alternate and for small values of α we have roughly

$$n \approx \frac{1}{2} f \approx \frac{1}{4} e^2 \approx \frac{1}{4} \epsilon^2 \quad .$$

Table I

$\frac{1}{f}$	α_0	$q_2(\alpha_0)$	$q'_2(\alpha_0)$
296.0	4° 42' 39.6"	.0000741879	.00270858
.1	36.8	1502	767
.2	33.9	1126	675
.3	31.0	0751	584
.4	28.2	0375	492
296.5	25.3	0000	401
.6	22.5	.0000739626	309
.7	19.6	9251	218
.8	16.8	8877	127
.9	13.9	8504	036
297.0	11.0	8130	.00269944
.1	8.2	7757	853
.2	5.3	7384	762
.3	2.5	7012	671
.4	4° 41' 59.7"	6640	581
297.5	56.8	6268	490
.6	54.0	5897	399
.7	51.1	5525	308
.8	48.3	5154	218
.9	45.4	4784	127
298.0	42.6	4414	037
.1	39.8	4044	.00268946
.2	36.9	3674	856
.3	34.1	3305	765
.4	31.3	2936	675
298.5	28.4	2567	585

Table Ia

Spheroid	α_0	$q_2(\alpha_0)$	$q'_2(\alpha_0)$
Clarke, 1880	4° 43' 52.8"	.0000751524	.00273203
Clarke, 1866	4° 43' 09.0"	745739	271798
Bessel, 1841	4° 41' 10.0"	730167	267998

Table II

Values of α

$\Delta r \backslash \Psi$	0°	45°	90°
0	4° 42' 11.0"	4° 42' 11.0"	4° 42' 11.0"
10	4° 41' 44.5"	4° 41' 44.5"	4° 41' 44.6"
20	4° 41' 18.0"	4° 41' 18.1"	4° 41' 18.2"
30	4° 40' 51.6"	4° 40' 51.7"	4° 40' 51.9"
40	4° 40' 25.3"	4° 40' 25.5"	4° 40' 25.6"
50	4° 39' 59.1"	4° 39' 59.3"	4° 39' 59.5"
60	4° 39' 32.9"	4° 39' 33.2"	4° 39' 33.4"
70	4° 39' 06.8"	4° 39' 07.2"	4° 39' 07.5"
80	4° 38' 40.9"	4° 38' 41.2"	4° 38' 41.6"
90	4° 38' 15.0"	4° 38' 15.3"	4° 38' 15.7"
100	4° 37' 49.1"	4° 37' 49.6"	4° 37' 50.0"

Values of $q_2(\alpha)$

$\Delta r \backslash \Psi$	0°	45°	90°
0	.0000738130	.0000738130	.0000738130
10	4659	4665	4671
20	1210	1222	1233
30	727783	727800	727817
40	4377	4400	4422
50	0992	1020	1048
60	717628	717662	717695
70	4285	4324	4363
80	0963	1007	1051
90	707661	707711	707760
100	4380	4435	4489

Values of $q_2'(\alpha)$

$\Delta r \backslash \Psi$	0°	45°	90°
0	.00269944	.00269944	.00269944
10	9097	9098	9100
20	8253	8256	8259
30	7413	7418	7422
40	6578	6583	6589
50	5746	5753	5760
60	4918	4926	4934
70	4094	4103	4113
80	3273	3284	3295
90	2457	2469	2481
100	1644	1658	1671

The values of Δr are given in kilometers. Note that at equator and pole Δr represents an altitude, but at geocentric latitude 45° it differs from an altitude, which incidentally might be variously defined, by a quantity of the order of $\frac{1}{2}f^2\Delta r$.

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